



University of Groningen

Effort- and flow-constraint reduction methods for structure preserving model reduction of port-Hamiltonian systems

Polyuga, Rostyslav V.; Schaft, Arjan J. van der

Published in:
Systems & Control Letters

DOI:
[10.1016/j.sysconle.2011.12.008](https://doi.org/10.1016/j.sysconle.2011.12.008)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2012

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Polyuga, R. V., & Schaft, A. J. V. D. (2012). Effort- and flow-constraint reduction methods for structure preserving model reduction of port-Hamiltonian systems. *Systems & Control Letters*, 61(3), 412-421. <https://doi.org/10.1016/j.sysconle.2011.12.008>

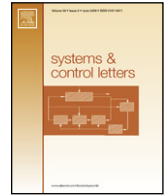
Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.



Effort- and flow-constraint reduction methods for structure preserving model reduction of port-Hamiltonian systems

Rostyslav V. Polyuga^{a,*}, Arjan J. van der Schaft^{b,1}

^a ABN AMRO N.V. Bank, Gustav Mahlerlaan 10, 1082PP (PAC HQ2015), Amsterdam, The Netherlands

^b Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O.Box 407, 9700 AK Groningen, The Netherlands

ARTICLE INFO

Article history:

Received 30 December 2010

Received in revised form

21 December 2011

Accepted 31 December 2011

Keywords:

Port-Hamiltonian systems

Structure preserving model reduction

Dirac structure

Effort-constraint method

Flow-constraint method

ABSTRACT

The geometric formulation of general port-Hamiltonian systems is used in order to obtain two structure preserving reduction methods. The main idea is to construct a reduced-order Dirac structure corresponding to zero power flow in some of the energy-storage ports. This can be performed in two canonical ways, called the effort- and the flow-constraint methods. We show how the effort-constraint method can be regarded as a projection-based model reduction method. Both the effort- and flow-constraint reduction methods preserve the stability and passivity properties of the original system, as a consequence of preserving the port-Hamiltonian structure.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

A standard way to model large-scale physical systems is *network modeling*. In this approach, the overall system is decomposed into (possibly many) interconnected subsystems. Network modeling has many advantages, such as reusability of subsystem models (libraries), flexibility (coarse models of subsystems may be replaced by more refined ones, leaving the rest of the system modeling untouched), hierarchical modeling, and control (by adding new subsystems as control components). In *port-based network modeling* (e.g., bond graph modeling), the overall system is decomposed into subsystems which are interconnected to each other through (vector) pairs of variables, whose product is the power exchanged among the subsystems. This approach is especially useful for the systematic modeling of *multi-physics systems*, where the subsystems belong to different physical domains (mechanical, electrical, hydraulic, etc.).

Since the beginning of the nineties of the previous century, it has been realized [1–5] that the mathematical models arising from port-based network modeling have an insightful geometric structure, which can be regarded as a generalization of the

geometric formulation of analytical mechanics into its Hamiltonian form. These geometric dynamical system models that follow directly from port-based network modeling have been called *port-Hamiltonian systems* [1,3,6].

The state-space dimensions of mathematical models arising from network modeling easily become very large; think, for example, of electrical circuits, multi-body systems, or spatial discretization of distributed-parameter systems. Thus, there is an immediate need for *model reduction* methods. However, since we want the reduced order models again to be interconnectable to other (sub-) systems, we want to retain the port-Hamiltonian structure of the reduced order systems. Furthermore, we want to preserve structural properties, such as energy conservation, passivity and existence of conservation laws as implied by the port-Hamiltonian structure. Thus, the problem arises of *structure preserving model reduction* of port-Hamiltonian systems.

The geometric formulation of port-Hamiltonian systems motivates a model reduction approach for general port-Hamiltonian systems (possibly also including the algebraic constraints), which involves the construction of a reduced order Dirac structure, and subsequently the construction of a reduced Hamiltonian. This approach is directly based on port-based modeling by replacing interconnections with almost zero energy flow by zero-power constraints. In this paper we treat two canonical structure preserving model reduction methods, called the effort-constraint reduction method and the flow-constraint reduction method. We show how the effort-constraint method in suitable coordinates can be regarded as a projection-based model reduction method. We

* Corresponding author. Tel.: +31 20 6280573.

E-mail addresses: rostyslav.polyuga@gmail.com, rostyslav.polyuga@nl.abnamro.com (R.V. Polyuga), A.J.van.der.Schaft@rug.nl (A.J. van der Schaft).

¹ Tel.: +31 50 3633731, +31 50 3633379.

suggest these coordinates for the effort-constraint method and balanced coordinates for both the effort- and flow-constraint methods as a possible choice of the coordinate system in order to obtain the reduced order models.

Structure preserving model reduction of port-Hamiltonian systems was also studied in [7–9]. The perturbation approach is considered in [10,11]. The use of the (rational) Krylov methods is addressed in [12–16], see also [17]. A recent overview of port-Hamiltonian model reduction methods can be found in [18]. Model reduction of nonlinear port-Hamiltonian systems is discussed in [19,20]. Model reduction of general nonlinear systems is gaining more attention in recent years, see for example [21–24] and references therein. For a general overview of model reduction methods, we refer the reader to [25–27].

Preliminary results of this work are presented in [28].

The paper is organized as follows. The general definition of port-Hamiltonian systems using the notion of a Dirac structure is given in Section 2. In Section 3, we explain the idea behind structure preserving model reduction based on zero-power constraints. Equational representations of the reduced order models are given in Section 4. These equational representations give rise to the effort- and flow-constraint reduced models for linear input-state-output port-Hamiltonian systems in Section 5. A numerical example, presented in Section 6, illustrates the performance of the effort- and flow-constraint reduction methods.

2. Dirac structures and port-Hamiltonian systems

The first main ingredient in the definition of a port-Hamiltonian system is the notion of a Dirac structure, which relates the power variables of the composing elements of the system in a power-conserving manner. The power variables always appear in conjugated pairs (such as voltages and currents, or generalized forces and velocities), and therefore mathematically they are modeled to take their values in dual linear spaces.

Definition 1 ([29]). Let \mathcal{F} be a linear space with a dual space $\mathcal{E} := \mathcal{F}^*$, and a duality product denoted as $\langle e | f \rangle \in \mathbb{R}$, with $f \in \mathcal{F}$ and $e \in \mathcal{E}$. In vector notation, we simply write the duality product as $\langle e | f \rangle = e^T f$. We call \mathcal{F} the space of *flow* variables, and $\mathcal{E} = \mathcal{F}^*$ the space of *effort* variables. Define on $\mathcal{F} \times \mathcal{E}$ the following indefinite bilinear form

$$\langle (f_1, e_1), (f_2, e_2) \rangle = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle.$$

A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a constant² *Dirac structure* if $\mathcal{D} = \mathcal{D}^\perp$, where \mathcal{D}^\perp is the orthogonal complement of \mathcal{D} with respect to the indefinite bilinear form $\langle \cdot, \cdot \rangle$.

Remark 1. It can be shown [29,26] that in the case of a finite-dimensional linear space \mathcal{F} , a Dirac structure \mathcal{D} is equivalently characterized as a subspace such that $e^T f = \langle e | f \rangle = 0$ for all $(f, e) \in \mathcal{D}$, together with $\dim \mathcal{D} = \dim \mathcal{F}$. The property $\langle e | f \rangle = 0$ for all $(f, e) \in \mathcal{D}$ corresponds to power conservation.

A *port-Hamiltonian system* is defined as follows. We start with a Dirac structure \mathcal{D} (see Fig. 1) on the space of all flow and effort variables involved:

$$\mathcal{D} \subset \mathcal{F}_x \times \mathcal{E}_x \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_p \times \mathcal{E}_p. \quad (1)$$

The space $\mathcal{F}_x \times \mathcal{E}_x$ is the space of flow and effort variables corresponding to the *energy-storing* elements (to be defined later on), the space $\mathcal{F}_R \times \mathcal{E}_R$ denotes the space of flow and effort variables

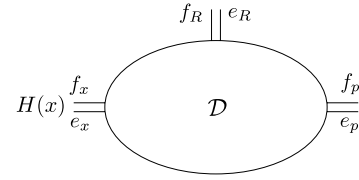


Fig. 1. Geometric definition of a port-Hamiltonian system.

of the *resistive* elements, while $\mathcal{F}_p \times \mathcal{E}_p$ is the space of flow and effort variables corresponding to the external *ports* (or sources). The property $\langle e | f \rangle = 0$ for all $(f, e) \in \mathcal{D}$ implies that the power supplied through the external port is distributed between the energy-storing port and the resistive port.

The vector of all the flow and effort variables of a port-Hamiltonian system

$$\begin{aligned} f_x &\in \mathcal{F}_x, & e_x &\in \mathcal{E}_x, & f_R &\in \mathcal{F}_R, \\ e_R &\in \mathcal{E}_R, & f_p &\in \mathcal{F}_p, & e_p &\in \mathcal{E}_p \end{aligned}$$

is required to be in the Dirac structure

$$(f_x, e_x, f_R, e_R, f_p, e_p) \in \mathcal{D}. \quad (2)$$

The constitutive relations for the *energy-storing* elements are defined as follows. Let the Hamiltonian $H : \mathcal{X} \rightarrow \mathbb{R}$ denote the total energy of the energy-storing elements with state variables $x = (x_1, x_2, \dots, x_n)^T$; i.e., the total energy is given as $H(x)$. In the sequel, we will take $\mathcal{X} = \mathcal{F}_x$.³ Then the energy-storage constitutive relations are given as⁴

$$\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x). \quad (3)$$

This immediately implies the following energy balance

$$\frac{d}{dt} H = -e_x^T f_x, \quad (4)$$

that is, the increase in total energy $H(x)$ is equal to the *power* $-e_x^T f_x$ provided to the energy-storing elements.

The constitutive relations for the *resistive elements* are given as⁵

$$f_R = -\varphi(e_R), \quad (5)$$

for some function φ satisfying

$$e_R^T \varphi(e_R) > 0 \quad \text{for all } e_R \neq 0. \quad (6)$$

Linear resistive elements are given as

$$f_R = -R e_R, \quad R = R^T \geq 0. \quad (7)$$

The interpretation is that power is always *dissipated* by the resistive elements.

Definition 2. Consider a Dirac structure (1), a Hamiltonian $H : \mathcal{X} \rightarrow \mathbb{R}$ with constitutive relations (3), and a resistive relation $f_R = -\varphi(e_R)$ as in (5). Then the dynamics (2) of the resulting port-Hamiltonian system is given as

$$\left(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), -\varphi(e_R(t)), e_R(t), f_p(t), e_p(t) \right) \in \mathcal{D}. \quad (8)$$

³ This can be immediately generalized to taking \mathcal{X} to be an n -dimensional manifold with tangent space being \mathcal{F}_x .

⁴ The vector $\frac{\partial H}{\partial x}(x)$ of partial derivatives of H will throughout be denoted as a column vector.

⁵ This can be immediately generalized to a nonlinear resistive relation $R(f_R, e_R) = 0$ having the property that $e_R^T f_R \leq 0$ for all f_R, e_R satisfying this relation.

² For the general definition of Dirac structures on manifolds we refer to e.g., [29,2].

It follows [3,6] from the power-conservation property of Dirac structures, and (4) and (6) that

$$\frac{d}{dt}H = -e_R^T \varphi(e_R) + e_P^T f_P \leq e_P^T f_P, \quad (9)$$

thus showing *passivity* if the Hamiltonian H is bounded from below.

3. Structure preserving model reduction based on power conservation

Consider a general port-Hamiltonian system (8), with state variables x and total stored energy $H(x)$. Let us assume that we have been able to find (e.g., by some balancing technique) a splitting of the state-space variables $x = (x_1^T, x_2^T)^T$, $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, having the property that the x_2 coordinates hardly contribute to the input–output behavior of the system, and thus could be omitted from the state-space description. It is easily seen that the usual *truncation* method for obtaining a reduced order model in the reduced state x_1 in general does *not* preserve the port-Hamiltonian structure, like it does not preserve the passivity property, see e.g., [25,18, Remark 2.12]. The same holds for the so-called singular perturbation reduction method, as was mentioned in [18, Remark 2.14]; see also [30,31].

In which way is it possible to retain the port-Hamiltonian structure in model reduction? Recall that in the definition of a port-Hamiltonian system, the vector of flow and effort variables (2) is required to be in the Dirac structure

$$(f_x^1, f_x^2, e_x^1, e_x^2, f_R, e_R, f_P, e_P) \in \mathcal{D}, \quad (10)$$

while the flow and effort variables f_x, e_x are linked to the constitutive relations of the energy-storage by

$$\begin{aligned} \dot{x}_1 &= -f_x^1, & \frac{\partial H}{\partial x_1}(x_1, x_2) &= e_x^1, \\ \dot{x}_2 &= -f_x^2, & \frac{\partial H}{\partial x_2}(x_1, x_2) &= e_x^2, \end{aligned}$$

which is shown in Fig. 2. This figure is a zoomed-in version of Fig. 1. The basic idea of structure preserving model reduction considered in this paper is to “cut” the interconnection

$$\dot{x}_2 = -f_2, \quad \frac{\partial H}{\partial x_2}(x_1, x_2) = e_2, \quad (11)$$

between the energy storage corresponding to x_2 and the Dirac structure, in such a way that no energy is transferred. Hence the exchange of energy between the energy storage and the other system elements through the Dirac structure happens only via the port associated to x_1 , with x_1 being the reduced order state vector.

The energy flow through the interconnection (11) is set equal to zero by making both power products

$$\left(\frac{\partial H}{\partial x_2} \right)^T \dot{x}_2 \quad \text{and} \quad (e_2)^T f_2$$

equal to zero.

This can be done in the two following canonical ways (see also [28])

(i): Set

$$\frac{\partial H}{\partial x_2}(x_1, x_2) = 0, \quad e_2 = 0. \quad (12)$$

The first equation imposes an algebraic constraint on the space variables $x = (x_1^T, x_2^T)^T$. Under general conditions on the Hamiltonian H , this constraint allows one to solve for x_2 as a function of x_1 : $x_2 = x_2(x_1)$, leading to a *reduced Hamiltonian*

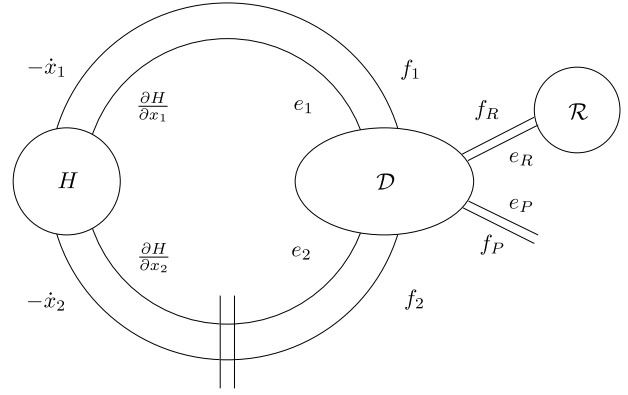


Fig. 2. Model reduction scheme.

$$H_{\text{red}}^{\text{ec}}(x_1) := H(x_1, x_2(x_1)).$$

Furthermore, the second equation defines the *reduced Dirac structure*⁶

$$\begin{aligned} \mathcal{D}_{\text{red}}^{\text{ec}} := \{ & (f_x^1, e_x^1, f_R, e_R, f_P, e_P) \mid \exists f_2 \text{ such that} \\ & (f_x^1, e_x^1, f_2, 0, f_R, e_R, f_P, e_P) \in \mathcal{D} \}, \end{aligned}$$

leading to the reduced port-Hamiltonian system

$$\left(-\dot{x}_1, \frac{\partial H_{\text{red}}^{\text{ec}}}{\partial x_1}(x_1), -\varphi(e_R), e_R, f_P, e_P \right) \in \mathcal{D}_{\text{red}}^{\text{ec}}.$$

We will call this reduction method the *effort-constraint* reduction method, since it constrains the efforts e_2 and $\frac{\partial H}{\partial x_2}$ to zero.

(ii): Set

$$\dot{x}_2 = 0, \quad f_2 = 0. \quad (13)$$

The first equation imposes the constraint

$$x_2 = c,$$

where the constant c can be taken to be zero, and thus defines the reduced Hamiltonian

$$H_{\text{red}}^{\text{fc}}(x_1) := H(x_1, c), \quad (14)$$

while the second equation leads to the reduced Dirac structure

$$\begin{aligned} \mathcal{D}_{\text{red}}^{\text{fc}} := \{ & (f_x^1, e_x^1, f_R, e_R, f_P, e_P) \mid \exists e_2 \text{ such that} \\ & (f_x^1, e_x^1, 0, e_2, f_R, e_R, f_P, e_P) \in \mathcal{D} \}, \end{aligned} \quad (15)$$

and the corresponding reduced port-Hamiltonian system

$$\left(-\dot{x}_1, \frac{\partial H_{\text{red}}^{\text{fc}}}{\partial x_1}(x_1), -\varphi(e_R), e_R, f_P, e_P \right) \in \mathcal{D}_{\text{red}}^{\text{fc}}. \quad (16)$$

We call this approach the *flow-constraint* reduction method, because it constrains the flows $-\dot{x}_2, f_2$.

An important open question, which will not be answered in this paper, is how to choose the coordinates $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in such a way that the energy flow between the energy storage corresponding to x_2 and the rest of the system through the Dirac structure is very small (negligible) at all time instants. Then the approximations (12) and (13) are at least from an energy transfer point of view well justified.

In Section 5.5 we will briefly discuss the closely related question of how to choose the coordinates in such a manner that the reduced model is close to the full order model from an input–output point of view.

⁶ $\mathcal{D}_{\text{red}}^{\text{ec}}$ is the composition of the full order Dirac structure \mathcal{D} with the Dirac structure on the space of flow and effort variables f_2, e_2 defined by $e_2 = 0$. It is proven in [32] that $\mathcal{D}_{\text{red}}^{\text{ec}}$ is indeed a Dirac structure.

4. Equational representations of the reduced order models

We will now provide explicit equational representations of the above two methods for structure preserving model reduction starting from the general representation by DAEs of the full order model (for details see [1–3,6,18]):

$$F_x \dot{x} = E_x \frac{\partial H}{\partial x}(x) - F_R \varphi(e_R) + E_R e_R + F_P f_P + E_P e_P, \quad (17)$$

where the matrices $F_x, E_x, F_R, E_R, F_P, E_P$ satisfy [3,6]

$$\begin{aligned} E_x F_x^T + F_x E_x^T + E_R F_R^T + F_R E_R^T + E_P F_P^T + F_P E_P^T &= 0, \\ \text{rank} \begin{bmatrix} F_x & F_R & F_P & E_x & E_R & E_P \end{bmatrix} &= n_x + n_R + n_P, \end{aligned} \quad (18)$$

with $n_x = \dim \mathcal{F}_x$, $n_R = \dim \mathcal{F}_R$, $n_P = \dim \mathcal{F}_P$.

Corresponding to the splitting of the state vector x into $x = (x_1^T, x_2^T)^T$, $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, where r is the dimension chosen for the reduced order model, and the respective splitting of the flow and effort vectors f_x, e_x into f_x^1, f_x^2 and e_x^1, e_x^2 , we write

$$F_x = \begin{bmatrix} F_x^1 & F_x^2 \end{bmatrix}, \quad E_x = \begin{bmatrix} E_x^1 & E_x^2 \end{bmatrix}. \quad (19)$$

Proposition 1. The reduced Dirac structure $\mathcal{D}_{\text{red}}^{\text{ec}}$ corresponding to the effort-constraint $e_x^2 = 0$ is given by the explicit equations

$$\begin{aligned} L^{\text{ec}} F_x^1 f_x^1 + L^{\text{ec}} E_x^1 e_x^1 + L^{\text{ec}} F_R f_R + L^{\text{ec}} E_R e_R \\ + L^{\text{ec}} F_P f_P + L^{\text{ec}} E_P e_P = 0, \end{aligned} \quad (20)$$

where L^{ec} is any matrix of maximal rank satisfying

$$L^{\text{ec}} F_x^2 = 0. \quad (21)$$

Proof. For the proof of the statement, we refer the reader to [32]. \square

Similarly, the following result holds true.

Proposition 2. The reduced Dirac structure $\mathcal{D}_{\text{red}}^{\text{fc}}$ corresponding to the flow-constraint $f_x^2 = 0$ is given by the equations

$$L^{\text{fc}} F_x^1 f_x^1 + L^{\text{fc}} E_x^1 e_x^1 + L^{\text{fc}} F_R f_R + L^{\text{fc}} E_R e_R + L^{\text{fc}} F_P f_P + L^{\text{fc}} E_P e_P = 0, \quad (22)$$

where L^{fc} is any matrix of maximal rank satisfying

$$L^{\text{fc}} E_x^2 = 0. \quad (23)$$

Proof. The proof can be found again in [32]. \square

It follows that the reduced order model resulting from applying the effort-constraint method is given by

$$\begin{aligned} L^{\text{ec}} F_x^1 \dot{x}_1 &= L^{\text{ec}} E_x^1 \frac{\partial H_{\text{red}}^{\text{ec}}}{\partial x_1}(x_1) - L^{\text{ec}} F_R \varphi(e_R) \\ &\quad + L^{\text{ec}} E_R e_R + L^{\text{ec}} F_P f_P + L^{\text{ec}} E_P e_P, \end{aligned} \quad (24)$$

whereas the reduced order model resulting from applying the flow-constraint method is given by

$$\begin{aligned} L^{\text{fc}} F_x^1 \dot{x}_1 &= L^{\text{fc}} E_x^1 \frac{\partial H_{\text{red}}^{\text{fc}}}{\partial x_1}(x_1) - L^{\text{fc}} F_R \varphi(e_R) \\ &\quad + L^{\text{fc}} E_R e_R + L^{\text{fc}} F_P f_P + L^{\text{fc}} E_P e_P. \end{aligned} \quad (25)$$

The steps of model reduction leading to the reduced order models (24), (25) are depicted in Fig. 3. First, we consider a full order port-Hamiltonian system with the corresponding full order Dirac structure. Second, we reduce the full order Dirac structure to obtain the reduced order Dirac structure. Finally, given the reduced order Dirac structure, we obtain the reduced order system. At the same time, we are approximating the full order Hamiltonian of the full order model in order to obtain the reduced order Hamiltonian of the reduced order model. Note that the reduced order models obtained in this way are port-Hamiltonian by construction.

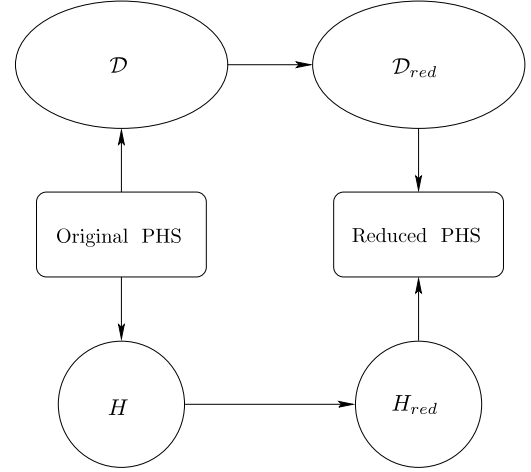


Fig. 3. Steps of model reduction of a full order port-Hamiltonian system (PHS) with the Hamiltonian H and the Dirac structure \mathcal{D} .

5. Reduced models for linear input-state-output port-Hamiltonian systems

In this section we specialize the results of the previous section to the case of linear input-state-output port-Hamiltonian systems [3,6]

$$\begin{cases} \dot{x} = (J - R)Qx + Gu, & J = -J^T, R = R^T \geq 0, Q = Q^T, \\ y = G^T Qx. \end{cases} \quad (26)$$

The model (26) is obtained after the termination of the resistive port. In order to use the Dirac structure representation (17) of this model, we rewrite (26) in the form

$$\begin{cases} \dot{x} = JQx + G_R f_R + Gu, \\ y = G^T Qx, \\ e_R = G_R^T Qx, \quad f_R = -\bar{R}e_R, \end{cases} \quad (27)$$

where the matrix \bar{R} is such that

$$G_R \bar{R} G_R^T = R. \quad (28)$$

Splitting of the state vector into $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, for r being the dimension of the reduced order model, then leads to the following partitioned system description

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_{R1} \\ G_{R2} \end{bmatrix} f_R \\ \quad + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u, \\ y = \begin{bmatrix} G_1^T & G_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ e_R = \begin{bmatrix} G_{R1}^T & G_{R2}^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f_R = -\bar{R}e_R. \end{cases} \quad (29)$$

5.1. Effort-constraint method

Rewriting these equations into the form (17), and applying the general effort-constraint reduction method (20) from the previous section, yields (assuming that Q_{22} is invertible) the reduced order port-Hamiltonian model (30).

Proposition 3. The effort-constraint reduction method (20) results in the following reduced order port-Hamiltonian model

$$\begin{cases} \dot{x}_1 = (J_{11} - R_{11})(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + G_1 u, \\ y_{\text{ec}} = G_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 \end{cases} \quad (30)$$

for the full order model (26).

Proof. Full details for the derivation of the reduced order model (30) are relegated to Appendix A, see also Proposition 1. \square

The reduced model (30) was already obtained by direct methods in [8], as well as in scattering coordinates in [9].

5.2. Flow-constraint method

The application of the flow-constraint method (22) to (29) (rewritten in the DAE-form (17)) is more involved. Assuming invertibility of J_{22} , the flow-constraint method is seen to lead to the reduced order port-Hamiltonian model (31).

Proposition 4. The flow-constraint reduction method (22) results in the following reduced order port-Hamiltonian model

$$\begin{cases} \dot{x}_1 = [(J_s - \beta^T Z_{sk} \beta) - \beta^T Z_{sym} \beta] Q_{11} x_1 \\ \quad + [(-\alpha^T + \beta^T Z_{sk} \gamma^T) - (-\beta^T Z_{sym} \gamma^T)] u, \\ y_{fc} = [(-\alpha - \gamma Z_{sk} \beta) - \gamma Z_{sym} \beta] Q_{11} x_1 \\ \quad + [(-\eta + \gamma Z_{sk} \gamma^T) + \gamma Z_{sym} \gamma^T] u, \end{cases} \quad (31)$$

where we have adopted the notation

$$\begin{aligned} \alpha &:= G_2^T J_{22}^{-1} J_{21} - G_1^T, & \beta &:= G_{R_2}^T J_{22}^{-1} J_{21} - G_{R_1}^T, \\ \gamma &:= G_2^T J_{22}^{-1} G_{R_2}, & \delta &:= G_{R_2}^T J_{22}^{-1} G_{R_2}, \\ \eta &:= G_2^T J_{22}^{-1} G_2, & Z &:= \bar{R}(I - \delta \bar{R})^{-1}, \\ Z_{sym} &:= \frac{1}{2}(Z + Z^T), & Z_{sk} &:= \frac{1}{2}(Z - Z^T), \\ J_s &:= J_{11} - J_{12} J_{22}^{-1} J_{21}. \end{aligned} \quad (32)$$

Proof. Full details can be found in Appendix B, see also Proposition 2. \square

Note that even though we started with a full order port-Hamiltonian system (26) without feed-through terms, the flow-constraint method, in contrast to the effort-constraint method, results in the reduced order model (31), which is a linear input-state-output port-Hamiltonian system with feedthrough terms [6]⁷:

$$\begin{cases} \dot{x}_1 = (J_r - R_r) Q_r x_1 + (G_r - P_r) u, \\ y_{fc} = (G_r^T + P_r^T) Q_r x_1 + (M_r + S_r) u, \end{cases}$$

where the reduced order matrices are

$$\begin{aligned} J_r &= J_s - \beta^T Z_{sk} \beta, & R_r &= \beta^T Z_{sym} \beta, \\ Q_r &= Q_{11}, & G_r &= -\alpha^T + \beta^T Z_{sk} \gamma^T, \\ P_r &= -\beta^T Z_{sym} \gamma^T, & M_r &= -\eta + \gamma Z_{sk} \gamma^T, \\ S_r &= \gamma Z_{sym} \gamma^T. \end{aligned}$$

One can easily verify that J_r , M_r are skew-symmetric, R_r , S_r are positive semi-definite symmetric, Q_r is positive definite symmetric, while R_r , P_r and S_r satisfy

$$\begin{bmatrix} R_r & P_r \\ P_r^T & S_r \end{bmatrix} \geq 0$$

(Lemma 1 in Appendix B demonstrates that Z_{sym} in (32) is positive definite).

⁷ See [6] for an extensive discussion on port-Hamiltonian systems with feed-through terms.

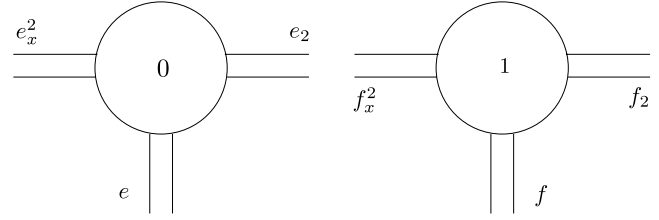


Fig. 4. 0-junction (left) and 1-junction (right).

Remark 2. Whenever $G_2 = 0$, then the reduced order port-Hamiltonian system (31) specializes to the reduced order system without feed-through terms

$$\begin{cases} \dot{x}_1 = [J_s - (G_{R_1} - J_{12} J_{22}^{-1} G_{R_2}) Z (G_{R_1}^T \\ \quad - G_{R_2}^T J_{22}^{-1} J_{21})] Q_{11} x_1 + G_1 u, \\ y_{fc} = G_1^T Q_{11} x_1. \end{cases} \quad (33)$$

Remark 3. In the case of a lossless full order port-Hamiltonian system (26), that is $R = 0$ and $\bar{R} = 0$, the reduced order port-Hamiltonian system (31) is also lossless and is given as

$$\begin{cases} \dot{x}_1 = J_s Q_{11} x_1 + (G_1 - J_{12} J_{22}^{-1} G_2) u, \\ y_{fc} = (G_1^T - G_2^T J_{22}^{-1} J_{21}) Q_{11} x_1 - G_2^T J_{22}^{-1} G_2 u. \end{cases} \quad (34)$$

5.3. Effort- and flow-constraint methods in the bond-graph modeling framework

Effort- and flow-constraint methods have a direct interpretation from the bond-graph modeling point of view. Constraining the efforts

$$e_x^2 = \frac{\partial H}{\partial x_2}(x_1, x_2) = 0, \quad e_2 = 0,$$

in the lower part of Fig. 2, which results in the effort-constraint method, corresponds to the so-called 0-junction with constraint $e = 0$, shown in Fig. 4 (without orientations). On the other hand, constraining the flows

$$f_x^2 = -\dot{x}_2 = 0, \quad f_2 = 0,$$

as in the flow-constraint method, corresponds to the 1-junction with constraint $f = 0$ (see again Fig. 4). The 0- and 1-junctions represent generalized, i.e., domain independent, Kirchhoff current and voltage laws, respectively, and are the common ways to model physical constraints in bond-graph modeling. For details see e.g., [6].

5.4. The effort-constraint method and moment matching

Consider a single-input single-output port-Hamiltonian system (26)

$$\begin{cases} \dot{x} = (J - R) Q x + g u, \\ y = g^T Q x, \end{cases} \quad (35)$$

with an input matrix $g \in \mathbb{R}^{n \times 1}$. The effort-constraint method from Proposition 3, which leads in this case to the following reduced order model

$$\begin{cases} \dot{x}_1 = F_{11} (Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) x_1 + g_1 u, \\ y_{ec} = g_1^T (Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) x_1, \end{cases} \quad (36)$$

turns out to have a relation to the projection based methods matching moments of the full order system at certain points in the complex plane. The moment-matching approach, discussed in [25]

and the references therein, requires computing (e.g., by the Arnoldi procedure) a map $V_r \in \mathbb{R}^{n \times r}$, $x = V_r x_r$, with $x_r \in \mathbb{R}^r$ being the reduced order state vector. Then the map V_r is used to project the full order system (35) in such a way that r moments of (35) and the projected reduced order system match at $s_0 \in \mathbb{C}$ or at infinity. The moment-matching approach for port-Hamiltonian systems is presented in [12,15,13,14,16,17] with an overview in [18].

To illustrate the relation of the effort-constraint method to moment matching, consider a full order single-input single-output port-Hamiltonian system (35). The co-energy variable representation of (35) (with the usual coordinate transformation $e = Qx$, see [6,18]) will take the form

$$\begin{cases} \dot{e} = Q(J - R)e + Qgu, \\ y = g^T e. \end{cases} \quad (37)$$

Recall from the literature on moment matching (see again [25]) that a map $V_e \in \mathbb{R}^{n \times r}$ matches the first r moments of (37) at infinity or at $s_0 \in \mathbb{C}$ if (for $A := Q(J - R)$)

$$\text{im} V_e = \text{im}[Qg : AQg : \dots : A^{r-1}Qg], \quad \text{or} \quad (38)$$

$$\text{im} V_e = \text{im}[(A - s_0 I)^{-1}Qg : \dots : (A - s_0 I)^{-r}Qg],$$

respectively. Then the following result holds true.

Theorem 1. Suppose that the energy coordinates x of (35) are such that the projection map

$$V_e = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}, \quad \text{with } V_1 \in \mathbb{R}^{r \times r} \text{ invertible}, \quad (39)$$

matches the first r moments at $s_0 \in \mathbb{C}$ or at infinity of the full order system in co-energy coordinates (37). Then the reduced order port-Hamiltonian model obtained by the effort-constraint method

$$\begin{cases} \dot{x}_1 = F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + g_1u, \\ y_{ec} = g_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1, \end{cases} \quad (40)$$

matches the first r moments of the full order system (35) at $s_0 \in \mathbb{C}$ or at infinity.

Proof. The moment matching projection of the rewritten port-Hamiltonian system (37)

$$\begin{cases} Q^{-1}\dot{e} = (J - R)e + gu, \\ y = g^T e, \end{cases}$$

is given by

$$\begin{cases} V_e^T Q^{-1} V_e \dot{e}_r = V_e^T (J - R) V_e e_r + V_e^T gu, \\ \hat{y} = g^T V_e e_r. \end{cases} \quad (41)$$

Using the well-known matrix inversion formula, we get

$$V_e^T Q^{-1} V_e = \begin{bmatrix} V_1^T & 0 \end{bmatrix} \begin{bmatrix} Q_s^{-1} & * \\ * & * \end{bmatrix} \begin{bmatrix} V_1 \\ 0 \end{bmatrix} = V_1^T Q_s^{-1} V_1,$$

where $Q_s = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$ is the Schur complement of Q . Therefore the reduced order system becomes

$$\begin{cases} V_1^T Q_s^{-1} V_1 \dot{e}_r = V_1^T (J_{11} - R_{11}) V_1 e_r + V_1^T g_1 u, \\ \hat{y} = g_1^T V_1 e_r. \end{cases}$$

Since $e = V_e e_r$ implies that $e_1 = V_1 e_r$ and since V_1^T is invertible, the reduced order model transforms to

$$\begin{cases} Q_s^{-1} \dot{e}_1 = (J_{11} - R_{11}) e_1 + g_1 u, \\ \hat{y} = g_1^T e_1, \end{cases}$$

which is, after the transformation from co-energy to energy coordinates $e_1 = Q_s x_1$, nothing but the reduced order system (40) obtained by the effort-constraint method

$$\begin{cases} \dot{x}_1 = F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + g_1 u, \\ y_{ec} = g_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1. \end{cases} \quad (42)$$

Since there are only linear coordinate transformations involved, the moments of (42) and (41), and thus (42) and (35) are the same, which completes the proof. \square

5.5. The choice of the coordinate system for model reduction

As already indicated before, we do not address in this paper the question of how to choose the coordinate system in which we apply either the effort- or the flow-constraint method. One possible choice of coordinates is balanced coordinates using Lyapunov balancing, positive real (Chapter 4 of [18]) or some other type of balancing. Another choice for the flow-constraint method would be to choose the coordinates where $G_2 = 0$, which would significantly simplify the expression of the reduced order model (31), see (33). The effort-constraint method for the SISO port-Hamiltonian systems naturally suggests coordinates x as in Theorem 1, in order to match moments at specific points in the complex plane, which would pose a question of how to find such coordinates in a numerically efficient way.

6. Numerical example

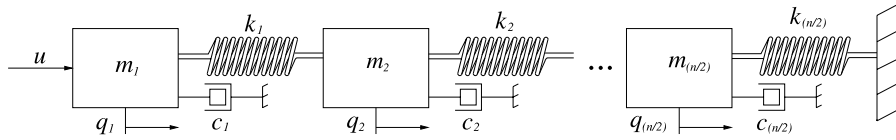
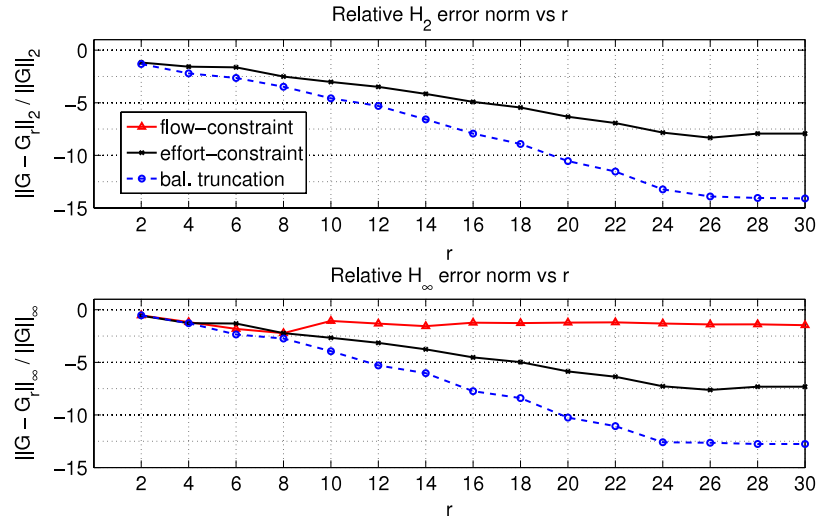
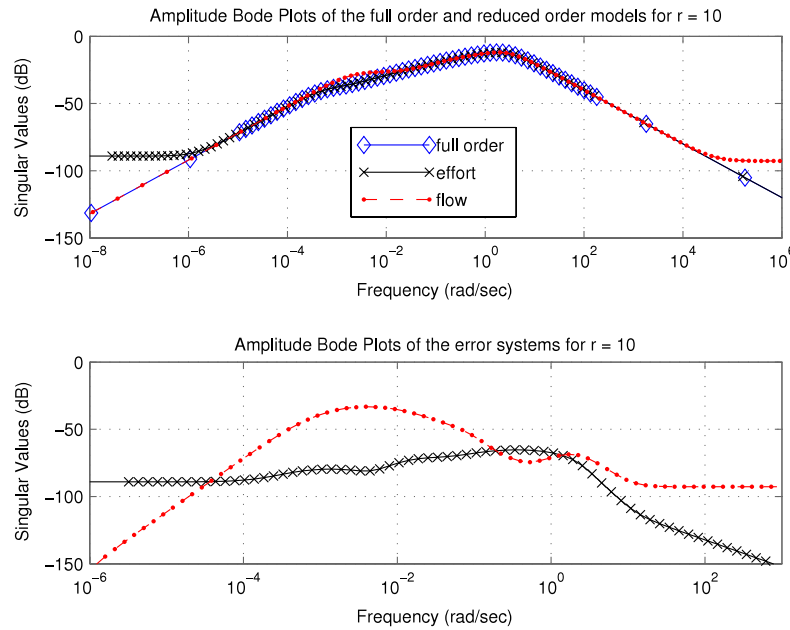
Consider an n -dimensional full order port-Hamiltonian mass-spring-damper system as shown in Fig. 5, with masses m_i , spring constants k_i and damping constants $c_i \geq 0$, for $i = 1, \dots, n/2$. p_i and q_i are the momentum and displacement of the mass m_i , respectively. The external force acting on the first mass, m_1 , is the input u , while its velocity is the output y . State variables are defined in the following way: for $i = 1, \dots, n/2$, $x_{2i-1} = q_i$ and $x_{2i} = p_i$. A detailed port-Hamiltonian description of this system is given in [13].

We considered a 100-dimensional mass-spring-damper system with $m_i = 1$, $k_i = 2$, and $c_i = 3.6$, and applied the effort-constraint method from (30), the flow-constraint method as in (31) and the regular balanced truncation. The coordinates chosen for reduction are (Lyapunov) balanced coordinates.

The reduced order systems are constructed for the orders $r = 2$ to $r = 30$ with increments of 2. Evolution of the relative \mathcal{H}_2 - and \mathcal{H}_∞ -norms is shown in Fig. 6. The figure demonstrates that both relative norms for the effort-constraint method consistently decay as the dimension of the reduced order models increases, perhaps apart from the orders $r = 28$ and $r = 30$. The relative \mathcal{H}_∞ -norm for the flow-constraint method surprisingly does not show similar decaying behavior. Therefore the effort-constraint method outperforms the flow-constraint method for the considered mass-spring-damper system for all dimension of the reduced order models except for $r = 6$. The performance of the effort-constraint method was also studied in [8,13,18]. Note that a feed-through term is present in the flow-constraint method (31). Thus, the \mathcal{H}_2 -norms of the flow-constraint method are unbounded and are not shown in the figure.

The regular balanced truncation method, as seen from Fig. 6, outperforms the presented effort- and flow-constraint methods for all dimensions of the reduced order models. Yet we want to underline that the balanced truncation method does not preserve the port-Hamiltonian structure (as explained in [18, Remark 2.12]).

The amplitude Bode plots of the full, reduced and error systems for $r = 10$ are shown in Fig. 7. The figure exhibits that the approximation by the flow-constraint method is better for low

Fig. 5. n -dimensional mass-spring-damper system.Fig. 6. Evolution of the relative \mathcal{H}_2 - and \mathcal{H}_∞ -norms.Fig. 7. Amplitude Bode plots for $r = 10$.

frequencies, while the approximation by the effort-constraint method does a better job for high frequencies. The error plot illustrates that the \mathcal{H}_∞ -norm is larger for the reduced order model by the flow-constraint method. This is consistent with the information from Fig. 6.

Naturally, the considered reduced order models, produced by the effort- and flow-constraint methods, inherit the port-Hamiltonian structure, are asymptotically stable and passive.

7. Conclusions

In this paper, we considered two port-Hamiltonian structure preserving model reduction methods: the effort-constraint method and the flow-constraint method. Both reduction methods preserve the stability and passivity properties of the original system, as a consequence of preserving the port-Hamiltonian structure. These methods arise from the geometric description of

general port-Hamiltonian systems, and are based on the idea of replacing the interconnections to the energy-storage, which carry little power, by zero-power constraints. These constraints can be interpreted within the bond-graph modeling framework as effort- or flow-constraints. We showed that the effort-constraint method, applied in particular coordinates, matches the first moments of the SISO full order port-Hamiltonian system at specific points in the complex plane. A numerical example illustrates the performance of the effort- and flow-constraint methods. A systematic way of choosing the coordinates for the full order port-Hamiltonian system in order to obtain the most accurate approximation from the input-output point of view is an important question left for future research. Relation of the flow-constraint method to moment matching methods, as well as possible error bounds for the effort- and flow-constraint methods, are additional open questions.

Appendix A. Effort-constraint reduction

Consider the full order port-Hamiltonian system (29) with a splitting of the state according to the dimension r chosen for the reduced order model:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_{R1} \\ G_{R2} \end{bmatrix} f_R + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u, \\ y = \begin{bmatrix} G_1^T & G_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ e_R = \begin{bmatrix} G_{R1}^T & G_{R2}^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f_R = -\bar{R}e_R. \end{cases} \quad (43)$$

The full order Dirac structure corresponding to the model (43) is given by the explicit equation in the DAE form (17)

$$F_x \dot{x} = E_x \frac{\partial H}{\partial x}(x) + F_R f_R + E_R e_R + F_P f_P + E_P e_P, \quad (44)$$

or

$$\begin{bmatrix} I_n \\ 0_{m \times n} \\ 0_{m_R \times n} \end{bmatrix} \dot{x} = \begin{bmatrix} J \\ -G^T \\ -G_R^T \end{bmatrix} \frac{\partial H}{\partial x}(x) + \begin{bmatrix} G_R \\ 0_{m \times m_R} \\ 0_{m_R} \end{bmatrix} f_R + \begin{bmatrix} 0_{n \times m_R} \\ 0_{m \times m_R} \\ I_{m_R} \end{bmatrix} e_R + \begin{bmatrix} G \\ 0_{m \times m} \\ 0_{m_R \times m} \end{bmatrix} f_P + \begin{bmatrix} 0_{n \times m} \\ I_m \\ 0_{m_R \times m} \end{bmatrix} e_P,$$

where m_R is the dimension of the vector of resistive variables f_R , e_R , and m is that of the vectors of input and output variables $f_P = u$, $e_P = y$.

Using the notation $e_x = \frac{\partial H}{\partial x}(x)$, the above equation reads

$$\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \\ 0_{m \times n} \\ 0_{m_R \times n} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ -G_1^T & -G_2^T \\ -G_{R1}^T & -G_{R2}^T \end{bmatrix} \begin{bmatrix} e_x^1 \\ e_x^2 \end{bmatrix} + \begin{bmatrix} G_{R1} \\ G_{R2} \\ 0_{m \times m_R} \\ 0_{m_R} \end{bmatrix} f_R + \begin{bmatrix} 0_{n \times m_R} \\ 0_{m \times m_R} \\ I_{m_R} \end{bmatrix} e_R + \begin{bmatrix} G_1 \\ G_2 \\ 0_{m \times m} \\ 0_{m_R \times m} \end{bmatrix} u + \begin{bmatrix} 0_{n \times m} \\ I_m \\ 0_{m_R \times m} \end{bmatrix} y. \quad (45)$$

Recall from Section 4 that the effort-constraint method assumes finding a (non-unique) maximal rank matrix L^{ec} satisfying

$$L^{\text{ec}} F_x^2 = 0,$$

as well as setting $e_x^2 = 0$. The simplest choice for L^{ec} is

$$L^{\text{ec}} = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_{m_R} \end{bmatrix}. \quad (46)$$

Premultiplying (45) with L^{ec} , while setting $e_x^2 = 0$, leads to

$$\begin{bmatrix} I_r \\ 0_{m \times r} \\ 0_{m_R \times r} \end{bmatrix} \dot{x}_1 = \begin{bmatrix} J_{11} \\ -G_1^T \\ -G_{R1}^T \end{bmatrix} e_x^1 + \begin{bmatrix} G_{R1} \\ 0_{m \times m_R} \\ 0_{m_R} \end{bmatrix} f_R + \begin{bmatrix} 0_{r \times m_R} \\ 0_{m \times m_R} \\ I_{m_R} \end{bmatrix} e_R + \begin{bmatrix} G_1 \\ 0_{m \times m} \\ 0_{m_R \times m} \end{bmatrix} u + \begin{bmatrix} 0_{r \times m} \\ I_m \\ 0_{m_R \times m} \end{bmatrix} \hat{y}, \quad (47)$$

which is the equational representation (20)

$$L^{\text{ec}} F_x^1 f_x^1 + L^{\text{ec}} E_x^1 e_x^1 + L^{\text{ec}} F_R f_R + L^{\text{ec}} E_R e_R + L^{\text{ec}} F_P f_P + L^{\text{ec}} E_P e_P = 0,$$

of the reduced order Dirac structure (note that $f_x^1 = -\dot{x}_1$).

Recall from [18, Section 2.6.1] that setting $e_x^2 = 0$ implies that $e_x^1 = Q_s x_1$, where $Q_s = Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}$ is the Schur complement of the energy matrix Q . The equational representation (47) is hence equivalent to

$$\begin{cases} \dot{x}_1 = J_{11} Q_s x_1 + G_{R1} f_R + G_1 u, \\ \hat{y} = G_1^T Q_s x_1, \\ e_R = G_{R1}^T Q_s x_1. \end{cases} \quad (48)$$

This is the reduced order port-Hamiltonian model by the effort-constraint method with the open resistive port. Termination of the resistive port employing the original linear relation $f_R = -\bar{R}e_R$ (while using $R_{11} = G_{R1} \bar{R} G_{R1}^T$ after the corresponding splitting of R from (28)) leads to the reduced order port-Hamiltonian model by the effort-constraint method (30)

$$\begin{cases} \dot{x}_1 = (J_{11} - R_{11})(Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) x_1 + G_1 u, \\ y_{\text{ec}} = G_1^T (Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) x_1. \end{cases} \quad (49)$$

Appendix B. Flow-constraint reduction

We start with the equational representation of the full order Dirac structure (45). A maximal rank matrix L^{fc} satisfying

$$L^{\text{fc}} E_x^2 = 0$$

is

$$L^{\text{fc}} = \begin{bmatrix} I_r & -J_{12} J_{22}^{-1} & 0 & 0 \\ 0 & G_2^T J_{22}^{-1} & I_m & 0 \\ 0 & G_{R2}^T J_{22}^{-1} & 0 & I_{m_R} \end{bmatrix}, \quad (50)$$

assuming that J_{22} is invertible (for even dimensions J_{22} is necessarily invertible). For details, see again Section 4. Premultiplication of the Eqs. (45) with L^{fc} and setting $f_x^2 = -\dot{x}_2 = 0$ leads to the following equational representation of the reduced order Dirac structure

$$\begin{bmatrix} I_r & -J_{12} J_{22}^{-1} \\ 0_{m \times r} & G_2^T J_{22}^{-1} \\ 0_{m_R \times r} & G_{R2}^T J_{22}^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} J_s & 0 \\ G_2^T J_{22}^{-1} J_{21} - G_1^T & 0 \\ G_{R2}^T J_{22}^{-1} J_{21} - G_{R1}^T & 0 \end{bmatrix} \begin{bmatrix} e_x^1 \\ e_x^2 \end{bmatrix} + \begin{bmatrix} G_{R1} - J_{12} J_{22}^{-1} G_{R2} \\ G_2^T J_{22}^{-1} G_{R2} \\ G_{R2}^T J_{22}^{-1} G_{R2} \end{bmatrix} f_R + \begin{bmatrix} 0_{r \times m_R} \\ 0_{m \times m_R} \\ I_{m_R} \end{bmatrix} e_R + \begin{bmatrix} G_1 - J_{12} J_{22}^{-1} G_2 \\ G_2^T J_{22}^{-1} G_2 \\ G_{R2}^T J_{22}^{-1} G_2 \end{bmatrix} u + \begin{bmatrix} 0_{r \times m} \\ I_m \\ 0_{m_R \times m} \end{bmatrix} \hat{y}.$$

Using the notation as in (32)

$$\begin{aligned} \alpha &:= G_2^T J_{22}^{-1} J_{21} - G_1^T, & \beta &:= G_{R2}^T J_{22}^{-1} J_{21} - G_{R1}^T, \\ \gamma &:= G_2^T J_{22}^{-1} G_{R2}, & \delta &:= G_{R2}^T J_{22}^{-1} G_{R2}, \\ \eta &:= G_2^T J_{22}^{-1} G_2, \end{aligned}$$

the above equation takes the form

$$\begin{bmatrix} I_r \\ 0_{m \times r} \\ 0_{m_R \times r} \end{bmatrix} \dot{x}_1 = \begin{bmatrix} J_s \\ \alpha \\ \beta \end{bmatrix} e_x^1 + \begin{bmatrix} -\beta^T \\ \gamma \\ \delta \end{bmatrix} f_R + \begin{bmatrix} 0_{r \times m_R} \\ 0_{m \times m_R} \\ I_{m_R} \end{bmatrix} e_R \\ + \begin{bmatrix} -\alpha^T \\ \eta \\ -\gamma^T \end{bmatrix} u + \begin{bmatrix} 0_{r \times m} \\ I_m \\ 0_{m_R \times m} \end{bmatrix} \hat{y}. \quad (51)$$

The equational representation (51) of the reduced order Dirac structure implies the reduced order port-Hamiltonian model

$$\begin{cases} \dot{x}_1 &= J_s e_x^1 - \beta^T f_R - \alpha^T u, \\ \hat{y} &= -\alpha e_x^1 - \gamma f_R - \eta u, \\ 0 &= \beta e_x^1 + \delta f_R + e_R - \gamma^T u. \end{cases} \quad (52)$$

The resistive relation $f_R = -\bar{R}e_R$ allows to solve the third equation for e_R . This, after substituting the expression for e_R in the other equations, and using the fact that e_x^1 is such that $e_x^1 = Q_{11}x_1$ ($\dot{x}_2 = 0$ implies $x_2 = \text{constant}$ taken to be zero), results in the reduced order port-Hamiltonian model (31)

$$\begin{cases} \dot{x}_1 &= (J_s - \beta^T Z \beta) Q_{11} x_1 + (-\alpha^T + \beta^T Z \gamma^T) u, \\ y_{fc} &= (-\alpha - \gamma Z \beta) Q_{11} x_1 + (-\eta + \gamma Z \gamma^T) u, \end{cases} \quad (53)$$

where $Z = \bar{R}(I - \delta \bar{R})^{-1}$.

Note that the matrix L^{fc} from (50) is non-unique, as discussed before. Another possible choice for maximal rank L^{fc} , which leads to the same result (53), is given in Appendix B.2 of [18, p. 143].

Next, we prove that the symmetric part of the matrix Z is positive-definite, showing that the reduced order model obtained by the flow-constraint method is indeed port-Hamiltonian.

Lemma 1. Consider the matrix Z from (32) given as

$$Z := \bar{R}(I - \delta \bar{R})^{-1}$$

for a skew-symmetric matrix $\delta = -\delta^T = G_{R_2}^T J_{22}^{-1} G_{R_2}$, and a symmetric positive definite matrix $\bar{R} = \bar{R}^T > 0$. Then the matrix Z can be decomposed into its symmetric Z_{sym} and skew-symmetric Z_{sk} parts as follows:

$$Z_{\text{sym}} = (\bar{R}^{-1} - \delta \bar{R} \delta)^{-1}, \quad Z_{\text{sk}} = (\bar{R}^{-1} \delta^{-1} \bar{R}^{-1} - \delta)^{-1}.$$

Furthermore, the symmetric part of the matrix Z is positive definite:

$$Z_{\text{sym}} = (\bar{R}^{-1} - \delta \bar{R} \delta)^{-1} > 0.$$

Proof. The matrix Z can be rewritten as $Z = (\bar{R}^{-1} - \delta)^{-1}$. Then straightforward calculations show that

$$\begin{aligned} Z_{\text{sym}} &= \frac{1}{2}(Z + Z^T) \\ &= \frac{1}{2}[(\bar{R}^{-1} - \delta)^{-1} + (\bar{R}^{-1} + \delta)^{-1}] \\ &= \frac{1}{2}(\bar{R}^{-1} - \delta)^{-1}[(\bar{R}^{-1} + \delta) + (\bar{R}^{-1} - \delta)](\bar{R}^{-1} + \delta)^{-1} \\ &= (\bar{R}^{-1} - \delta)^{-1} \bar{R}^{-1} (\bar{R}^{-1} + \delta)^{-1} \\ &= (\bar{R}^{-1} - \delta)^{-1} (I + \delta \bar{R})^{-1} \\ &= [(I + \delta \bar{R})(\bar{R}^{-1} - \delta)]^{-1} \\ &= (\bar{R}^{-1} - \delta \bar{R} \delta)^{-1}. \end{aligned}$$

Similarly

$$Z_{\text{sk}} = \frac{1}{2}(Z - Z^T) = (\bar{R}^{-1} \delta^{-1} \bar{R}^{-1} - \delta)^{-1}.$$

Moreover, $Z = (\bar{R}^{-1} - \delta)^{-1}$ implies that $Z^{-1} = \bar{R}^{-1} - \delta$. Hence, the symmetric part of Z^{-1} , which is \bar{R}^{-1} , is necessarily positive definite.

Since any real vector w can be written as $w = Z^{-1}v$ for a certain v , it follows that

$$w^T Z w = v^T Z^{-T} Z Z^{-1} v = v^T Z^{-T} v = v^T Z^{-1} v > 0.$$

This proves that the symmetric part of Z is positive definite. \square

Finally, note that in the case of a lossless full order port-Hamiltonian system $\bar{R} = 0$ and, consequently, $Z = 0$.

References

- [1] A.J. van der Schaft, B.M. Maschke, The Hamiltonian formulation of energy conserving physical systems with external ports, *Archiv für Elektronik und Übertragungstechnik* 49 (5–6) (1995) 362–371.
- [2] M. Dalsmo, A.J. van der Schaft, On representations and integrability of mathematical structures in energy-conserving physical systems, *SIAM Journal on Control and Optimization* 37 (1999) 54–91.
- [3] A.J. van der Schaft, L_2 -Gain and Passivity Techniques in Nonlinear Control, in: *Lect. Notes in Control and Information Sciences*, vol. 218, Springer-Verlag, Berlin, 1996, second revised and enlarged ed., Springer-Verlag, London, 2000 (Springer Communications and Control Engineering Series).
- [4] A.J. van der Schaft, Port-controlled Hamiltonian systems: towards a theory for control and design of nonlinear physical systems, *Journal of the Society of Instrument and Control Engineers of Japan (SICE)* 39 (2000) 91–98.
- [5] R. Ortega, A.J. van der Schaft, I. Mareels, B.M. Maschke, Putting energy back in control, *Control Systems Magazine* 21 (2001) 18–33.
- [6] The Geoplex Consortium, Modeling and Control of Complex Physical Systems: The Port-Hamiltonian Approach, Springer, Berlin, Heidelberg, 2009.
- [7] R.V. Polyuga, A.J. van der Schaft, Model reduction of port-Hamiltonian systems as structured systems, in: *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, Budapest, Hungary, 5–9 July, 2010, pp. 1509–1513.
- [8] R.V. Polyuga, A.J. van der Schaft, Structure preserving port-Hamiltonian model reduction of electrical circuits, in: P. Benner, M. Hinze, J. ter Maten (Eds.), *Model Reduction for Circuit Simulation*, in: *Lecture Notes in Electrical Engineering*, Springer-Verlag, Berlin, Heidelberg, 2011, pp. 241–260.
- [9] R.V. Polyuga, A.J. van der Schaft, Structure preserving model reduction of port-Hamiltonian systems, in: *Proceedings of the 18th International Symposium on Mathematical Theory of Networks and Systems*, Blacksburg, Virginia, USA, July 28–August 1, 2008. Publication available from: <http://sites.google.com/site/rostyslavpolyuga/publications>.
- [10] C. Hartmann, V.-M. Vulcanov, Ch. Schütte, Balanced truncation of linear second-order systems: a Hamiltonian approach, *Multiscale Modeling & Simulation* 8 (2010) 1348–1367.
- [11] C. Hartmann, Balancing of dissipative Hamiltonian systems, in: I. Troch and F. Breitenecker, (Eds.), *Proceedings of MathMod 2009*, Vienna, February 11–13, 2009, Number 35 in ARGESIM-Reports, Vienna Univ. of Technology, Vienna, Austria, 2009, pp. 1244–1255.
- [12] R.V. Polyuga, A.J. van der Schaft, Structure preserving model reduction of port-Hamiltonian systems by moment matching at infinity, *Automatica* 46 (2010) 665–672.
- [13] S. Gugercin, R.V. Polyuga, C.A. Beattie, A.J. van der Schaft, Interpolation-based \mathcal{H}_2 model reduction for port-Hamiltonian systems, in: *Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, PR, China, December 16–18, 2009, pp. 5362–5369.
- [14] S. Gugercin, R.V. Polyuga, C.A. Beattie, A.J. van der Schaft, Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems, *Automatica* (2012) (in press). Publication available from: [arxiv:1101.2348](https://arxiv.org/abs/1101.2348).
- [15] R.V. Polyuga, A.J. van der Schaft, Structure preserving moment matching for port-Hamiltonian systems: Arnoldi and Lanczos, *IEEE Transactions on Automatic Control* 56 (6) (2011) 1458–1462.
- [16] T. Wolf, B. Lohmann, R. Eid, P. Kotyczka, Passivity and structure preserving order reduction of linear port-Hamiltonian systems using Krylov subspaces, *European Journal of Control* 16 (4) (2010) 401–406.
- [17] R.V. Polyuga, Discussion on: “Passivity and structure preserving order reduction of linear port-Hamiltonian systems using Krylov subspaces”, *European Journal of Control* 16 (4) (2010) 407–409.
- [18] R.V. Polyuga, Model reduction of port-Hamiltonian systems, Ph.D. Thesis, University of Groningen, 2010.
- [19] K. Fujimoto, Balanced realization and model order reduction for port-Hamiltonian systems, *Journal of System Design and Dynamics* 2 (3) (2008) 694–702. (special issue on) *Nonlinear Dynamics in Mechanical Systems*.
- [20] J.M.A. Scherpen, A.J. van der Schaft, A structure preserving minimal representation of a nonlinear port-Hamiltonian system, in: *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, December 9–11, 2008, pp. 4885–4890.
- [21] K. Fujimoto, J.M.A. Scherpen, Balanced realization and model order reduction for nonlinear systems based on singular value analysis, *SIAM Journal on Control and Optimization* 48 (7) (2010) 4591–4623.
- [22] A. Astolfi, Model reduction by moment matching for linear and nonlinear systems, *IEEE Transactions on Automatic Control* 55 (2010) 2321–2336.

- [23] T.C. Ionescu, Balanced truncation for dissipative and symmetric nonlinear systems, Ph.D. Thesis, University of Groningen, 2009.
- [24] J. Sjöberg, Optimal control and model reduction of nonlinear DAE models, Ph.D. Thesis, Linköping University, 2008.
- [25] A.C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, SIAM, Philadelphia, 2005.
- [26] W.H.A. Schilders, H.A. van der Vorst, J. Rommes, *Model Order Reduction: Theory, Research Aspects and Applications*, in: ECMI Series on Mathematics in Industry, vol. 13, Springer-Verlag, Berlin, Heidelberg, 2008.
- [27] P. Benner, M. Hinze, J. ter Maten, *Model Reduction for Circuit Simulation*, in: *Lecture Notes in Electrical Engineering*, Springer-Verlag, Berlin, Heidelberg, 2011.
- [28] A.J. van der Schaft, R.V. Polyuga, Structure-preserving model reduction of complex physical systems, in: *Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, PR, China, December 16–18, 2009, pp. 4322–4327.
- [29] T.J. Courant, Dirac manifolds, *Transactions of the American Mathematical Society* 319 (1990) 631–661.
- [30] K. Fernando, H. Nicholson, Singular perturbational model reduction of balanced systems, *IEEE Transactions on Automatic Control* 27 (1982) 466–468.
- [31] M. Green, D.J.N. Limebeer, *Linear Robust Control*, Prentice-Hall, 1995.
- [32] J. Cervera, A.J. van der Schaft, A. Banos, Interconnection of port-Hamiltonian systems and composition of Dirac structures, *Automatica* 43 (2007) 212–225.